A geometrical approach to super W-induced gravities in two dimensions

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Abstract

A geometrical study of supergravity defined on $(1\,|\,1)$ complex superspace is presented. This approach is based on the introduction of generalized superprojective structures extending the notions of super Riemann geometry to a kind of super W-Riemann surfaces. On these surfaces a connection is constructed. The zero curvature condition leads to the super Ward identities of the underlying supergravity. This is accomplished through the symplectic form linked to the (super)symplectic manifold of all super gauge connections. The BRST algebra is also derived from the knowledge of the super W-symmetries which are the gauge transformations of the vector bundle canonically associated to the generalized superprojective structures. We obtain the possible consistent BRST (super)anomalies and their cocycles related by the descent equations. Finally we apply our considerations to the case of supergravity.

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1   Introduction

In 1985 Zamolodchikov introduced new symmetries in conformal models generated by currents of spin higher than 2, whose commutation relations were shown to have non linear terms. These new algebras, the W-algebras, were also shown to appear in integrable systems through Poisson brackets. They lead to induced (classical) W-gravities which are higher-spin gauge theories in two dimensions whose gauge algebras are these W-algebras just as the Virasoro algebra appears as the residual symmetry of gauge fixed gravity in two dimensions. For recent reviews see [1, 2]. It is natural to interpret such a class of conformal field theories as possible realizations of the W-geometries introduced in [3, 4, 5]. The approach of [3, 5] starts with the embedding of a 2-dim base manifold into a $n-1$ dimensional Kähler manifold whereas [4] is a recent development in the light-cone gauge. In this paper we generalize to the (1,1) supersymmetric case the geometrical setting given in [4].

The numerous links between 2D gravity and integrable systems through Poisson brackets are well known. Since the evolution of an integrable system can be thought of as a zero-curvature condition associated with some gauge group it seems natural to consider theories of the W_n induced gravities based on such a condition. The starting point of these theories is the vanishing condition of the field strength associated to a pair of matrices $(A_z, J_{\bar{z}})$, giving in the standard case the chiral Virasoro Ward-identity. After all, since the success of the Polyakov formulation [6] was to show that the unexpected $SL(2, \mathbb{R})$ current algebra arises in 2D gravity in the light-cone gauge, it is not astonishing that the first attempts use a group approach, the generalization to higher W-gravities consisting of replacing $SL(2, \mathbb{R})$ by some other non-compact real Lie group. More precisely the current $J_{\bar{z}}$ is parametrized as $J_{\bar{z}} = h^{-1}\bar{\partial}h$ whereas $A_z = g^{-1}\partial g$ where h, g are some group valued functions [7]. The matrix A_z contains the projective connection and the fields associated to it.

More recently Zucchini [4] has presented a formalism in which the usual Riemann surface is embedded in a n-dimensional complex manifold to which is canonically associated a $SL(n, \mathbb{C})$ fiber bundle. On this bundle a connection $\mathcal{A}$ with zero curvature is defined. This connection appears as a pair of matrices (Ω, Ω^*) which can be parametrized by introducing projective structures (μ_i, ρ_i) generalizing the pair $(\mu_{\bar{z}}{}^z, \rho_{zz})$ consisting of the Beltrami coefficient and the projective connection respectively on the ordinary Riemann surface. Working in this framework one finds $(n-1)$ pairs of generalized Beltrami differentials and projective connections characterizing a kind of "W_n" Riemann surface, which is assumed to be the geometrical way to reach the basic notions behind the W_n gravity theories i.e. the W_n-algebras. The geometric structure underlying these algebras appears as extra data on the Riemann surface. The zero curvature condition on the connection $\mathcal{A}$ is naturally ensured by the definition of its components in terms of a basic matrix W, namely $\Omega \equiv \partial W W^{-1}$, $\Omega^* \equiv \bar{\partial}W W^{-1}$. In the standard

cases (W_2, W_3) the expression of the resulting matrices (Ω, Ω^*) as functions of the gauge fields and the spin-s currents, respectively, is identical to the result of the group theory approach [8] when $h \equiv g$ (the connection $\mathcal{A}$ being a pure gauge). Moreover the local expression of h can be obtained in that case by taking a Gauss decomposition for h and the two formalisms coincide since $W = h^{-1}$, Zucchini's formalism providing a geometrical interpretation for the physical fields entering in the W-gravities.

The essential advantage of this approach is to define the W_n symmetries as gauge transformations of the flat vector bundle canonically associated to the generalized projective structures. From the knowledge of these symmetries the off-shell nilpotent BRST algebra for an arbitrary W_n model is derived [4]. Several other advantages can be emphasized: the generalization to arbitrary W_n models is automatic and is uniquely limited by technical complications; gluing properties of the fields under conformal coordinate changes are known "ab initio" and result from the formalism itself. Furthermore this formulation allows us to interpolate between various W theories thus taking into account their "nested" structure, $W_n \subset W_{n+1}$, where the inclusion symbol indicates that the formulation of W_n can be obtained from W_{n+1} by setting to zero the projective variables occuring at the level $n + 1$.

Most of the existing results, with some exceptions [9], concern the bosonic theory; complete studies in the supersymmetric case are still lacking. Accordingly, the systematic manifestly (1,1) supersymmetric extension of [4] presented here is an attempt to fill this gap. First we show (sect.2) that generalized superprojective structures may be parametrized in a one-to-one fashion by pairs of superfields which generalize the super-Beltrami differential and the superprojective connection. In supersymmetry, besides obvious technical difficulties, one has to face features which do not appear at the bosonic level. In particular, since some superfields involve only non physical fields, it seems natural to restrict the geometry by turning them off. When going to higher n the number of possibilities of this kind increases and the full model, although very cumbersome, can give birth to several meaningful and interesting developments.

The physical geometrical fields are not the fields (μ_i, ρ_i) which emerge naturally from the construction since in general these objects do not change in a homogeneous way under conformal coordinate transformations. In fact the physical fields $\Phi \equiv (\tilde{\mu}_i, \tilde{\rho}_i)$ are sections [1] ($\bar{k}k^{-p}$ with $1 \leq p \leq n - 1$ and k^q with $3 \leq q \leq n$) of the fiber bundle which transform as differentials: $\Phi_b = \Phi_a(k)^h(\bar{k})^{\bar{h}}$, h and $\bar{h}$ being the conformal weights of Φ. They appear as combinations of the (μ_i, ρ_i). A method exists to construct systematically the $\tilde{\rho}_i$ using the approach of [10]. Having obtained these fields by transposing these ideas to the supersymmetric case, we use the natural connection $\mathcal{A}$ to construct a symplectic form which provides a systematic way of obtaining the $\tilde{\mu}_i$ (sect.3).

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[1] except q=2 which concerns the particular case of the projective connection.

In sect.4, the BRST off-shell nilpotent algebra associated to an arbitrary super W_n induced gravity [2] (SW_n), is constructed from the SW_n symmetries which are defined as gauge transformations of the vector bundle canonically associated to the generalized superprojective structures. We present a general formulation of a consistent and covariant (i.e.well-defined on the super Riemann surface and obeying the Wess-Zumino consistency condition) super anomaly which may occur and of the cocycles linked to it by the BRST operator. In the last section we discuss the example of the induced supergravity (SW_2), writing the model in its full generality and comparing our results with existing ones.

For sake of clarity some details are collected in an appendix.

2　Geometrical setting

Starting from a $(1 \,|\, 1)$ complex superspace with coordinates (z, θ) we consider a supermanifold $\mathcal{M}$ which is obtained by patching together local coordinate charts $\{V_a, (z, \bar{z}, \theta, \bar{\theta})_a\}$. The basis of the tangent space is $(\partial_z, \partial_{\bar{z}}, D, \bar{D})$ where the super-derivatives are defined by

$$D = \partial_\theta + \theta \partial_z,$$

$$\bar{D} = \partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}}.$$

They obey

$$D^2 = \partial_z \equiv \partial \qquad \text{and} \qquad \bar{D}^2 = \partial_{\bar{z}} \equiv \bar{\partial}.$$

Under a change of reference structure $(z_a, \theta_a) \to (z_b, \theta_b)$ the vector field D transforms as

$$D_a = (D_a \theta_b) D_b + (D_a z_b - \theta_b D_a \theta_b) \partial_{z_b} + (D_a \bar{\theta}_b) \bar{D}_b + (D_a \bar{z}_b - \bar{\theta}_b D_a \bar{\theta}_b) \partial_{\bar{z}_b}.$$

The complex supermanifold thus defined becomes a $N = 1$ super Riemann surface (SRS) $S\Sigma$ if the transition functions $z_b(z_a, \bar{z}_a, \theta_a, \bar{\theta}_a)$, $\theta_b(z_a, \bar{z}_a, \theta_a, \bar{\theta}_a)$ (and their complex conjugates) between two local coordinate charts $(U_a, (z, \bar{z}, \theta, \bar{\theta})_a)$ and $(U_b, (z, \bar{z}, \theta, \bar{\theta})_b)$ satisfy the following conditions of super-conformality [3]

$$\bar{D}_a z_b = \bar{D}_a \theta_b = 0 \qquad \text{and} \qquad D_a z_b = \theta_b D_a \theta_b.$$

With these conditions the super-derivative transforms homogeneously. An atlas of superprojective coordinates on a SRS (without boundary) $S\Sigma$ is a collection of homeomorphisms $\{(Z, \Theta)_\alpha\}$ [4] of $S\Sigma$ into $\mathbb{C}^{1|1}$, locally defined on domains $\{K_\alpha\}$

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[2]Here the index n is choosen with reference to the underlying W_n model which is the bosonic limit of the super model considered.

[3]It is understood that the complex conjugate (cc) conditions are also to be taken into account.

[4]We will always use the Greek letters for the extended superprojective atlas and the Latin letters for the reference atlas on $S\Sigma$.

with the gluing laws on overlapping domains K_α and K_β [11]

$$
\begin{aligned}
Z_\beta &= \frac{mZ_\alpha + p}{qZ_\alpha + r} + \Theta_\alpha \frac{\gamma Z_\alpha + \delta}{(qZ_\alpha + r)^2} \\
\Theta_\beta &= \frac{\gamma Z_\alpha + \delta}{qZ_\alpha + r} + \Theta_\alpha \frac{1}{qZ_\alpha + r}(1 + \tfrac{1}{2}\gamma\delta)
\end{aligned}
\tag{1}
$$

where the matrix $\begin{pmatrix} m & p \\ q & r \end{pmatrix}$ belongs to $SL(2, \mathbb{C})$ whereas γ and δ are odd Grassmann numbers.

Such an atlas defines a supercomplex structure on $S\Sigma$, or equivalently, superconformal classes of metrics which are related to the reference structure (z, θ) by the super-Beltrami differentials through the super-Beltrami equations [11, 12, 13]. These structures are parametrized by two independent odd superfields $H_{\bar\theta}{}^z$ and $H_\theta{}^z$ (and the c.c. analogues). Since $H_\theta{}^z$ contains only auxiliary space-time fields, studies are in general limited to the special case $H_\theta{}^z = 0$, a restriction which is equivalent to $DZ = \Theta D\Theta$. The algebra which underlies this framework is the well-known super-Virasoro algebra.

Our approach to the super W-algebras is based on a straightforward generalization of the notion of superprojective coordinates. It consists in enlarging the set of these coordinates (Z, Θ) by considering a collection of local maps $\{(Z^1, \ldots, Z^n; \Theta^1, \ldots, \Theta^n)_\alpha\}$ of $S\Sigma$ into $\mathbb{C}^{1|1}$. These variables can be gathered in the vector

$$
\mathcal{Z} = \begin{pmatrix} 1 & Z^1 & \cdots & Z^n & \Theta^1 & \cdots & \Theta^n \end{pmatrix}^{st}
$$

where st is the supertranspose and the Z^i, Θ^i which are functions of $(z, \theta, \bar z, \bar\theta)$ have respectively an even and odd grassmannian character. We further impose the transition functions on overlapping domains K_α, K_β to be

$$
\mathcal{Z}_\beta^i = \frac{\sum_{j=0}^{2n} \Phi_{\beta\alpha}^{ij} \mathcal{Z}_\alpha^j}{(\Phi_{\beta\alpha}^0, \mathcal{Z}_\alpha)_E}
\tag{2}
$$

where $\Phi_{\beta\alpha}$ is a constant non singular $(2n + 1) \times (2n + 1)$ complex matrix of super-determinant 1 and $(\Phi_{\beta\alpha}^0, \mathcal{Z}_\alpha)_E$ is the euclidean scalar product between the first row (labelled 0) of $\Phi_{\beta\alpha}$ and the vector $\mathcal{Z}$. These transition functions can be regarded as a generalization of the superprojective (Möbius) transformations (1). We now define the matrix W_0 by (the dot marks the matrix product)

$$
W_0 = \mathcal{D} \cdot \mathcal{Z}^{st}
\tag{3}
$$

where $\mathcal{D}$ is the vector

$$
\mathcal{D} = \begin{pmatrix} 1 & \partial & \cdots & \partial^n & D & D\partial & \cdots & D\partial^{n-1} \end{pmatrix}^{st}.
$$

We divide each coordinate by the superdeterminant Δ of W_0 (assuming that the non-singularity condition $\Delta \neq 0$ holds everywhere on $S\Sigma$), thus defining the matrix W

$$W = \mathcal{D} \cdot (\frac{1}{\Delta} \mathcal{Z})^{st}. \tag{4}$$

The matrix we get is

$$W = \begin{pmatrix}
\frac{1}{\Delta} & \frac{Z^1}{\Delta} & \cdots & \frac{Z^n}{\Delta} & \frac{\Theta^1}{\Delta} & \cdots & \frac{\Theta^n}{\Delta} \\
\partial(\frac{1}{\Delta}) & \partial(\frac{Z^1}{\Delta}) & \cdots & \partial(\frac{Z^n}{\Delta}) & \partial(\frac{\Theta^1}{\Delta}) & \cdots & \partial(\frac{\Theta^n}{\Delta}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
\partial^n(\frac{1}{\Delta}) & \partial^n(\frac{Z^1}{\Delta}) & \cdots & \partial^n(\frac{Z^n}{\Delta}) & \partial^n(\frac{\Theta^1}{\Delta}) & \cdots & \partial^n(\frac{\Theta^n}{\Delta}) \\
D(\frac{1}{\Delta}) & D(\frac{Z^1}{\Delta}) & \cdots & D(\frac{Z^n}{\Delta}) & D(\frac{\Theta^1}{\Delta}) & \cdots & D(\frac{\Theta^n}{\Delta}) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
D\partial^{n-1}(\frac{1}{\Delta}) & D\partial^{n-1}(\frac{Z^1}{\Delta}) & \cdots & D\partial^{n-1}(\frac{Z^n}{\Delta}) & D\partial^{n-1}(\frac{\Theta^1}{\Delta}) & \cdots & D\partial^{n-1}(\frac{\Theta^n}{\Delta})
\end{pmatrix}$$
$$\tag{5}$$

We will study the transformation law of this matrix under both an extended superprojective transformation $\mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ and a change of the local coordinates $z_a \to z_b$ where $z = (z, \theta)$; it reads

$$W_{b\beta} = \mathcal{D}_b \cdot (\frac{1}{\Delta_{b\beta}} \mathcal{Z}_\beta)^{st}. \tag{6}$$

In order to express $W_{b\beta}$ in terms of $W_{a\alpha}$ we first note that the derivatives become

$$D_b = e^X D_a \tag{7}$$
$$\partial_b = e^{2X}(\partial_a + (D_a X)D_a) \tag{8}$$

where we have set $e^{-X} = D_a \theta_b$ (which is the canonical 1-cocycle of $S\Sigma$ [11]). Then it is straightforward to build the transformation matrix T_{ba} defined by:

$$\mathcal{D}_b = T_{ba} \cdot \mathcal{D}_a.$$

The determinant Δ of W_0 transforms as

$$\Delta_{b\beta} = e^{nX} \frac{\Delta_{a\alpha}}{(\Phi^0_{\beta\alpha}, \mathcal{Z}_\alpha)_E}.$$

From the definition (2) it follows readily that

$$\Phi_{\alpha\alpha} = \mathbf{1},$$

$$\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha} = \mathbf{1}.$$

Thus Φ defines a flat $sl(n+1 \mid n)$ vector bundle on $S\Sigma$ [5]. We then have :
$W_{b\beta} = T_{ba} K_{ba} W_{a\alpha} \Phi^{st}_{\beta\alpha}$ where K_{ba} has the form

$$
\begin{pmatrix} K_0 & 0 \\ K_1 & K_2 \end{pmatrix}
$$

with

- for $0 \le i \le n$ and $j \le i$

$$
(K_0)_{ij} = \begin{pmatrix} i \\ j \end{pmatrix} \partial^{i-j} e^{-nX}
$$

- for $n+1 \le i \le 2n$ and $j \le i$

$$
(K_1)_{ij} = \begin{pmatrix} i-n-1 \\ j \end{pmatrix} D\partial^{i-n-j-1} e^{-nX}
$$

- for $n+1 \le i \le 2n$ and $n+1 \le j \le i$

$$
(K_2)_{ij} = \begin{pmatrix} i-n-1 \\ j-n-1 \end{pmatrix} \partial^{i-j} e^{-nX}.
$$

From now on we set : $\Lambda_{ba} = T_{ba} K_{ba}$. This matrix can be viewed as a transition function of a bundle over $S\Sigma$, namely the jet bundle. We recall its definition. A n-jet of a field ψ of weight $\frac{n}{2}$ (i.e. of a section of the canonical bundle over $S\Sigma$) is the vector field

$$
j_n\psi = (\psi, \partial\psi, ..., \partial^n\psi, D\psi, ..., D\partial^{n-1}\psi)^{st}.
$$

Under a superconformal change of coordinates we have

$$
j_n\psi_b = \Lambda_{ba} j_n\psi_a.
$$

Thus writing

$$
\Phi^{\vee}_{\alpha\beta} = W^{-1}_{a\alpha} \Lambda_{ab} W_{b\beta}
$$

where

$$
\Phi^{\vee}_{\alpha\beta} = \Phi^{st}_{\alpha\beta}{}^{-1}
$$

is the dual bundle, amounts to saying that the jet bundle Λ and the flat $sl(n+1 \mid n)$ bundle Φ are equivalent. We can now define the two matrices

$$
\Omega = DW \cdot W^{-1} \tag{9}
$$

---

[5] In general one has $\Phi_{\alpha\alpha} = c_\alpha \mathbb{1}$ and $\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha} = k_{\alpha\beta\gamma}\mathbb{1}$, where c_α and $k_{\alpha\beta\gamma}$ are constants. However since sdet$(\Phi)=1$ the result above follows.

$$\Omega^* = \bar{D}W \cdot W^{-1}. \tag{10}$$

The crucial property of these matrices is their independence on the choice of the index α, i.e. on the choice of a chart in the super-projective atlas. These super-matrices are odd and transform as

$$\begin{aligned}
\Omega_b &= e^X[\tilde{\Lambda}_{ba}\Omega_a\Lambda_{ba}^{-1} + (D_a\Lambda_{ba})\Lambda_{ba}^{-1}] & (11) \\
\Omega_b^* &= e^{\bar{X}}\tilde{\Lambda}_{ba}\Omega_a^*\Lambda_{ba}^{-1} & (12)
\end{aligned}$$

where the tilde means that the odd blocks of the matrix acquire a minus sign. It is straightforward to verify that Ω and Ω^* satisfy the following relation

$$\bar{D}\Omega + D\Omega^* + \tilde{\Omega}\Omega^* + \tilde{\Omega}^*\Omega = 0. \tag{13}$$

Furthermore we have

$$\begin{aligned}
str\,\Omega &= 0 & (14) \\
str\,\Omega^* &= 0. & (15)
\end{aligned}$$

The three last equations indicate that Ω and Ω^* can be viewed as the two components of a flat $sl(n+1|n)$ connection on the jet bundle Λ.

Not all the coefficients of Ω and Ω^* are independent. In fact there are only $2n$ independent fields for each matrix. The peculiar structure itself of the matrix W (i.e. a Wronskian structure) entails the relations

$$DW_{ij} = \begin{cases} W_{i+n+1,k} & \text{for } 0 \le i \le n-1 \\ W_{i-n,k} & \text{for } n+1 \le i \le 2n \end{cases} \tag{16}$$

and therefore

$$\Omega_{ij} = \begin{cases} \delta_{i+n+1,j} & \text{for } 0 \le i \le n-1 \\ \delta_{i-n,j} & \text{for } n+1 \le i \le 2n \end{cases}$$

The only remaining coefficients are the $\Omega_{n,j}$. But due to the relation (14) we also have $\Omega_{n,n} = 0$ leaving us with $2n$ independent fields $\Omega_{n,j}$ $j \ne n$. Thus the matrix Ω, in this block grading, looks like

$$\Omega = \begin{pmatrix}
0 & \cdots & \cdots & 0 & 1 & & \\
\vdots & & & \vdots & & \ddots & \\
0 & \cdots & \cdots & 0 & & & 1 \\
\Omega_{n,0} & \cdots & \Omega_{n,n-1} & 0 & \Omega_{n,n+1} & \cdots & \Omega_{n,2n} \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0
\end{pmatrix} \tag{17}$$

As regards Ω^* it is easier to use a grading by diagonals [9, 14] which is moreover better suited for the following applications, in particular for dealing with the root

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vectors of the Lie algebras. Of course there are several ways to transform the matrix from the block to the diagonal grading. Here we impose that all the 1 entries in Ω be gathered on the first diagonal below the main one. The matrix which permits one to pass from block to diagonal grading is defined by

$$M_{diag} = P^{-1} M_{block} P \qquad \text{and} \qquad (P)_{ij} = \delta_{ip(j)}$$

where p is the permutation given by :

$$p(2k+1) = 2n - k \qquad p(2k) = n - k.$$

Thanks to this grading we can decompose $sl(n+1|n)$ in a sum of subspaces B_i, in which each subspace corresponds to a diagonal. We choose to number these subspaces with respect to the main diagonal which will be the 0th one. Positive numbered diagonals will be located on the upper triangular part of the matrices and negative ones on the lower part:

$$sl(n+1|n) = \bigoplus_{s=-2n}^{2n} B_s = B_- \oplus B_0 \oplus B_+.$$

In this new grading the matrix Ω has the form

$$\Omega = \begin{pmatrix} 0 & \rho_2 & \rho_3 & \cdots & \rho_{2n+1} \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \tag{18}$$

where the ρ_i field is the $\Omega_{n,k}$ of conformal weight $\theta^i (\rho_i \equiv \rho_{|\theta^i})$. We note that this form of Ω is reminiscent of the Polyakov's partial-gauge fixed connection [15].

The flatness condition (13) contains the $2n$ super-holomorphy conditions obeyed by the ρ_i. These conditions are in fact the Ward identities of the induced super W-gravity model underlying this geometrical framework [6]. Furthermore, when $\bar{D}\Omega = 0$, it allows us to determine the elements of Ω^* in terms of these ρ_i and of $2n$ other independent superfields $\mu_{i+1} \equiv \mu \mid_{\theta}^{\theta^i} \equiv \Omega^*_{i,0}$, $i \neq 0$. This set of relations can be treated in three groups corresponding to the components of Ω^* in B_-, B_0 and B_+. For the first group we solve iteratively the equations for $s = j - i$ going from $-2n$ to -2, with, for each value of s, j running from 0 to $2n + s$. This provides us with all the entries below the main diagonal. The second group concerns the elements of the 0th diagonal : they are obtained for $s = -1$ and j going from 0 to $2n-1$, with the help of the condition (14). At last $0 \leq s \leq 2n-1$

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[6]The association of a zero curvature condition to the formulation of induced W_n gravity and its interpretation as an anomaly equation are not new and can be found in numerous earlier works (see for instance [16],[17]).

with j running from $2n$ to s gives the elements above the main diagonal[7]. The results are

for $s = -2n, \cdots, -2$,

$$j = 0 \qquad \Omega^*_{-s,1} = (-1)^{-s+1}(\mu_{-s} + D\mu_{-s+1})$$

$$j = 1, \cdots, 2n+s \qquad \Omega^*_{j-s,j+1} = (-1)^{1-s}(D\Omega^*_{j-s,j} + \Omega^*_{j-s-1,j}) + (-1)^j\mu_{j-s+1}\rho_{j+1},$$

for $s = -1$,

$$\Omega^*_{0,0} = \sum_{p=1}^{n}(\mu_{2p+1}\rho_{2p} - D\Omega^*_{2p,2p-1})$$

$$\Omega^*_{j,j} = \sum_{p=2}^{j}(D\Omega^*_{p,p-1} + (-1)^{p+1}\mu_{p+1}\rho_p) + \Omega^*_{0,0} + D\mu_2,$$

for $s = 0, \cdots, 2n-1$,

$$j = 2n \qquad \Omega^*_{2n-1-s,2n} = (-1)^s\mu_{2n+1-s}\rho_{2n+1} - D\Omega^*_{2n-s,2n}$$

$$j = 2n-1, \cdots, s+1 \qquad \Omega^*_{j-s-1,j} = (-1)^{1-s}\Omega^*_{j-s,j+1} + (-1)^{j-s}\mu_{j-s+1}\rho_{j+1} - D\Omega^*_{j-s,j}$$

Ω^* in this diagonal grading looks as follows

$$\Omega^* = \begin{pmatrix} * & * & \cdots & * & * \\ \mu_2 & * & \cdots & * & * \\ \mu_3 & * & \cdots & * & * \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mu_{2n+1} & * & \cdots & * & * \end{pmatrix}, \tag{19}$$

where the stars stand for expressions in terms of the μ_i.

Thus to the family of generalized superprojective structures $\{(Z^1, \ldots, Z^n; \Theta^1, \ldots, \Theta^n)\}$ on $S\Sigma$ is canonically associated a set of $2n$ pairs of geometrical fields (μ_i, ρ_i) which can be viewed as the generalized super-Beltrami differentials and as the generalized projective connections (i.e. the backgrounds fields), respectively, in the same way that usual projective structures are parametrized by the Beltrami coefficient and the Schwarzian derivative. These sets of fields contain the supersymmetric extension $H_{\bar\theta}{}^z$ of the ordinary Beltrami coefficient [12] and the superprojective connection. In fact this parametrization of the generalized projective structures in terms of the (μ_i, ρ_i) is one-to-one. Indeed starting from the pairs (μ_i, ρ_i) and

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[7]We are there in the situation referred to in [18] where Ω is composed of a constant part (which will be named J_- in the following section) and a part which contains the fields ρ_i; moreover the algebra is graded and J_- has a definite degree, namely -1. This is why elementary calculations lead to the results and it is not necessary to use the more elaborated and more general method of [18].

defining the matrices above we can obtain generalized superprojective structures canonically associated to the μ_i's and the ρ_i's. The equation (13) can be viewed as an integrability condition for a linear system of partial differential equations [8]

$$(D - \Omega)U = 0 \qquad\qquad (\bar{D} - \Omega^*)U = 0, \qquad (20)$$

with the constraint

$$sdetU = 1. \qquad (21)$$

To parametrize the matrix U we consider a set of local sections (differentials) transforming homogeneously

$$\Psi = \left(\begin{array}{ccc} \Psi_0, & \cdots, & \Psi_{2n} \end{array} \right)$$

and define

$$U_{ij} = D^i \Psi_j \qquad 0 \leq i \leq 2n,$$

Then the first equation (20) is equivalent to a set of differential equations $\mathcal{L}\Psi_j = 0$, $0 \leq j \leq 2n$, where

$$\mathcal{L} = D^{2n+1} - \sum_{l=1}^{2n}(\Omega_{n,l}D^{2n-l}) \qquad (22)$$

is the Lax operator in the (n-reduced) super-KP hierarchy. Such a system admits $2n + 1$ linearly independent local solutions $(\Psi_0, ..., \Psi_{2n})$ which are normalized so that (21) holds. The non-uniqueness is reduced by imposing that the Ψ_i's satisfy the second equation (20), and the zero-curvature condition (13) is now regarded as the compatibility equation of the linear system (20). For instance let us assume that Ψ_{2n} is nowhere vanishing so that the maps $Z^{n-i} = \frac{\Psi_{2i}}{\Psi_{2n}}$ and $\Theta^{n-i} = \frac{\Psi_{2i+1}}{\Psi_{2n}}$ with $1 \leq i \leq n$ are well defined; it is then easy to verify that these maps satisfy eqns. (1,2).

3 Classical super W-algebras

To obtain the W-algebras we consider the independent fields appearing in the connection defined in the preceding section. These fields are in general not covariant under a super-conformal change of the coordinates $\boldsymbol{z} = (z, \theta)$. The first step is to make a basis change for Ω in order to find new fields that transform homogeneously [10]. For this purpose we shall extend to the supersymmetric case the method of [19] (see also [20] and [21]). Then thanks to the connection we define a symplectic form which allows us to covariantize the fields appearing in Ω^*.

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[8]Thus this geometrical construction provides us with a Lax pair for the super Ward identities as integrability conditions.

3.1 The Drinfeld and Sokolov method in the supersymmetric case

As noted in the preceding section, due to its form the matrix Ω can be thought of as the matrix appearing in a first order matrix differential equation. If we write $\hat{\mathcal{L}}F = (D - \Omega)F = 0$ with $F = (f_0, f_1, \ldots, f_{2n})^{st}$, we can eliminate the f_i's to obtain a differential operator of the form [14]

$$\mathcal{L} = D^{2n+1} - \sum_{k=1}^{2n} \rho_{k+1} D^{2n-k} \tag{23}$$

where $\rho_k = \Omega_{0,k-1}$. We mention here that this operator can be factorized as follows

$$\mathcal{L} = (D - D\Phi_{2n})(D - D(\Phi_{2n-1} - \Phi_{2n}))...(D - D(\Phi_1 - \Phi_2))(D - D\Phi_1)$$

leading to new fields Φ_i that obey Toda equations under the zero curvature condition (13). This field redefinition is a generalized Miura transformation [22].

Ω can be decomposed as $J_- + R$ where $J_- = diag_{-1}(1, ..., 1)$. J_- can be identified with the sum of the negative root vectors f_i of $sl(n+1|n)$ (see appendix). It can also be regarded as one of the generators of an $osp(1|2)$ algebra [23]. Indeed defining

$$J_+ = \sum_{i=1}^{r} \sum_{j=1}^{r} (C^{-1})_{ij} e_i$$

where C and $r = 2n$ are the Cartan matrix and the rank of $sl(n+1|n)$ respectively, and

$$H = \{J_+, J_-\} \qquad X_\pm = \frac{1}{2}\{J_\pm, J_\pm\}$$

we have the following commutation relations

$$\begin{array}{ll} [H, J_\pm] = \pm J_\pm & [H, X_\pm] = \pm 2X_\pm \\ [X_\pm, J_\mp] = \mp J_\pm & [X_\pm, J_\pm] = 0 \end{array} \tag{24}$$

$$[X_\pm, X_\mp] = -H.$$

The matrices H, $J_\pm$ and $X_\pm$ generate an $osp(1|2)$ subalgebra [24] and H allows us to characterize the diagonal grading since, as can be straightforwardly verified, for any M in B_i, we have [9, 14]

$$[H, M] = iM. \tag{25}$$

Going back to the operator $\mathcal{L}$ it is known [10, 19] that $\hat{\mathcal{L}}$ is not the only matrix operator leading to $\mathcal{L}$ (23). In fact every $\hat{\mathcal{L}}' = \tilde{Q}\hat{\mathcal{L}}Q^{-1}$ (where Q is a coordinate dependent upper triangular matrix with 1 entries on the main diagonal) will lead

to the same $\mathcal{L}$. Writing $Q = \mathbf{1} + \sum_{i \geq 1} Q_i$, $Q_i \in B_i$ we easily see that we can obtain

$$\hat{\mathcal{L}}' = D - J_- - R'$$

where $R' \in B_+$ (like $(\Omega - J_-)$) provided $\{Q_1, J_-\} = 0$, which leads to $Q_1 = 0$. This gauge freedom on $\hat{\mathcal{L}}$ is used to reparametrize $\mathcal{L}$ in terms of new fields $\tilde{\rho}_i$ which are covariant under a super-conformal change of coordinates. To get this new $\hat{\mathcal{L}}'$ following [10] we decompose every B_i as :

$$B_i = [J_-, B_{i+1}] \oplus V_i$$

where $[.,.]$ is the graded commutator. The map $[J_-, \cdot]$ from B_{i+1} to B_i is injective (for $sl(n+1|n)$), so that

$$dim\,[J_-, B_{i+1}] = dimB_i - 1$$

and

$$dimV_i = 1.$$

We now require the matrix R' to belong to $V = \oplus V_i$. There remains the task of choosing a convenient basis for the V_i's. To do this we select a matrix J_1 in V_1 and use its powers J_1^k as a basis for V_k (We note that this choice corresponds to the highest weight gauge of [25]). Unlike the bosonic case where the choice is canonical [19] we cannot choose J_+ as a basis in V_1 since $J_+ = [J_-, X_+] \notin V$. We thus have to impose constraints on J_1 so that the matrix operator $\hat{\mathcal{L}}'$ be written

$$\hat{\mathcal{L}}' = D - J_- - \sum_{k=1}^{2n} \tilde{\rho}_{k+1} J_1^k \tag{26}$$

where the new coefficients $\tilde{\rho}_k$ are covariant of weight $\frac{k}{2}$ [9] except $\tilde{\rho}_3$ which transforms as a projective connection, i.e. under an infinitesimal change of coordinates we have

$$\delta_\epsilon \hat{\mathcal{L}}' = \left[\chi, \hat{\mathcal{L}}'\right] = \frac{1}{2}(D^5\epsilon)J_1^2 + \sum_{k=1}^{2n} \left(\epsilon\partial + \frac{1}{2}(D\epsilon)D + \frac{k+1}{2}(\partial\epsilon)\right) \tilde{\rho}_{k+1} J_1^k \tag{27}$$

since under an infinitesimal superconformal change of coordinates

$$\begin{cases} z' &= z + \epsilon + \frac{1}{2}(D\epsilon)\theta \\ \theta' &= \theta + \frac{1}{2}(D\epsilon) \end{cases}$$

the super Schwarzian derivative $S = \frac{\partial^2 \theta'}{D\theta'} - 2\frac{(\partial\theta')(\partial D\theta')}{(D\theta')^2}$ transforms as

$$\delta_\epsilon S = \left(\epsilon\partial + \frac{1}{2}(D\epsilon)D + \frac{3}{2}(\partial\epsilon)\right) S + \frac{1}{2}(D^5\epsilon).$$

---

[9]A field of weight $\frac{h}{2}$ transforms as $\psi(z', \theta')(D\theta')^h = \psi(z, \theta)$ i.e. in infinitesimal form $\delta_\epsilon \psi = \left(\epsilon\partial + \frac{1}{2}(D\epsilon)D + \frac{h}{2}(\partial\epsilon)\right)\psi$

The coordinate dependent matrix χ generates the transformation and is such that when applied on a vector $F = (f_0, f_1, \ldots, f_{2n})^{st}$ the lowest component f_{2n} transforms as a covariant field of weight $-\frac{n}{2}$ (since $\mathcal{L}$ is covariant when applied on fields of weight $-\frac{n}{2}$ [19, 26]). Expanding χ on the $osp(1\,|\,2)$ basis we obtain the following constraints

$$[J_+, J_1^k] = 0, \quad [J_1^2, J_-] = -J_+.$$

The unique solution to these constraints (up to a sign) is

$$J_1 = diag_{+1}(n, 1, n-1, 2, ..., n)$$

which is an operator obtained in [14] and the diffeomorphism generating operator χ is found to be

$$-\chi = \frac{1}{2}\left((D\epsilon)(J_- + R) + (\partial\epsilon)H + (D^3\epsilon)J_+ + (D^4\epsilon)X_+ + 2\epsilon(J_- + \tilde{R})(J_- + R) + 2\epsilon(DR)\right).$$

3.2 Covariantization of the fields

The Drinfeld and Sokolov method thus leads to covariant fields $\tilde{\rho}_k$, $k \neq 3$. To obtain explicitly these $\tilde{\rho}_k$ and the matrix Q we just have to compare $\tilde{Q}\hat{\mathcal{L}}$ with $\hat{\mathcal{L}}'Q$ in every subspace B_i. For the first ones we have

$$\begin{aligned}
\rho_2 &= n(n+1)\tilde{\rho}_2 \\
\rho_3 &= \frac{1}{2}n(n+1)(\tilde{\rho}_3 + D\tilde{\rho}_2).
\end{aligned}$$

Since Q is expressed in terms of the fields ρ_k and their derivatives the gauged fields $\tilde{\rho}_k$ contain non-linear combinations of these fields. Under a change of coordinates the field $\tilde{\rho}_3$ transforms as a projective connection i.e.

$$\tilde{\rho}_3' = e^{3X}(\tilde{\rho}_3 + S) \tag{28}$$

where S is the super-Schwarzian derivative.

A change of basis for $\hat{\mathcal{L}}$ generates the following transformations

$$\begin{aligned}
\Omega' &= \tilde{Q}\Omega Q^{-1} + DQ \cdot Q^{-1} \tag{29} \\
\Omega^{*\prime} &= \tilde{Q}\Omega^* Q^{-1} + \bar{D}Q \cdot Q^{-1}. \tag{30}
\end{aligned}$$

To covariantize the μ_k fields we introduce the symplectic form [27]

$$\omega = \int_{S\Sigma} str(\delta\Omega^* \wedge \delta\tilde{\Omega}) \tag{31}$$

defined on the manifold $S\mathcal{M}$ of connection one-forms $\mathcal{A}$

$$\mathcal{A} = \Omega d\mathbf{z} + \Omega^* d\bar{\mathbf{z}}, \tag{32}$$

where $d\boldsymbol{z} = (dz \mid d\theta)$, the operator δ in (31) being the exterior derivative on $S\mathcal{M}$. The equation (13) is nothing but the flatness condition $\mathcal{F} = 0$ for $\mathcal{A}$. This symplectic 2-form is well defined on $S\Sigma$ since the operator δ is inert with respect to a coordinate change $\boldsymbol{z}_a \longrightarrow \boldsymbol{z}_b$; namely the gluing properties (11) become

$$\delta\Omega_b = e^X \tilde{\Lambda} \delta\Omega_a \Lambda^{-1} \tag{33}$$

$$\delta\Omega_b^* = e^{\bar{X}} \tilde{\Lambda} \delta\Omega_a^* \Lambda^{-1}. \tag{34}$$

Using the explicit expression of Ω and Ω^* (19) we can write

$$\omega = \int_{S\Sigma} \sum_{k=1}^{2n} \delta\mu_{k+1} \wedge \delta\rho_{k+1},$$

this replacement being equivalent to the Hamiltonian reduction of [16] and [28] so that $S\mathcal{M}$ is in fact the reduced manifold. Letting δ act on ρ_{k+1} expressed in terms of the $\tilde{\rho}_j$ leads to a polynomial which is linear in $\delta\tilde{\rho}_j$ and possibly in its derivatives $\delta D^l \tilde{\rho}_j$. Then, with the help of integrations by parts, we can factorize $\delta\tilde{\rho}_{k+1}$, getting in this way the expression of $\delta\tilde{\mu}_{k+1}$:

$$\omega = \int_{S\Sigma} \sum_{k=1}^{2n} \delta\mu_{k+1} \wedge \delta\rho_{k+1} = \int_{S\Sigma} \sum_{k=1}^{2n} \delta\tilde{\mu}_{k+1} \wedge \delta\tilde{\rho}_{k+1}.$$

Since the covariantized field $\tilde{\mu}_{k+1}$ does not depend explicitly on the superprojective coordinates, we obtain straightforwardly $\tilde{\mu}_{k+1}$ from $\delta\tilde{\mu}_{k+1}$ Thanks to the fact that the integrand in ω is well-defined on a SRS the $\tilde{\mu}_k$ transform as

$$\tilde{\mu}_k{}' = e^{\bar{X}} e^{(1-k)X} \tilde{\mu}_k.$$

An explicit example of this construction is given in sect. 5.1. Note that although $\tilde{\rho}_3$ transforms as a projective connection $\tilde{\mu}_3$ is covariant. Indeed when one applies the functional exterior derivative δ to (28) the term S does not contribute since $\delta S = 0$.

Let us now study the algebra formed by the $\tilde{\rho}_i$ and analyze its spin content. Considering the operator $\hat{\mathcal{L}}$ we immediately see that we have spins

$$(1, \frac{3}{2}, 2, \frac{5}{2}, ..., n, n + \frac{1}{2})$$

or equivalently considering the physical component expansion [10] $(\tilde{\rho}_{i0}, \tilde{\rho}_{i1})$

$$\left((1, \frac{3}{2}); (\frac{3}{2}, 2); (2, \frac{5}{2}); ...; (n, n + \frac{1}{2}); (n + \frac{1}{2}, n + 1) \right).$$

Obviously we are dealing with the $N = 2$ super W_{n+1} algebra since the spins naturally gather in $N = 2$ supermultiplets this fact being intimately related to

---
[10] we have : $\tilde{\rho}_i = \tilde{\rho}_{i0} + \theta\tilde{\rho}_{i1}$

the $sl(n + 1 \mid n)$ algebra. Now as it is well known [9, 29] we can restrict our connection form to belong to a subalgebra of $sl(n+1 \mid n)$ namely $osp(2m \pm 1 \mid 2m)$ (with $4m \pm 1 = 2n + 1$) and find a Chevalley basis of these subalgebras which gives the same expressions for the generators of the $osp(1 \mid 2)$ subalgebra (24) leading to a very convenient characterization of the basis vectors J_1^k that belong to $osp(2m \pm 1 \mid 2m)$. Indeed they are such that $k = 2, 3$ mod 4. The spin content is now restricted to

$$\left((\frac{3}{2}, 2); (2, \frac{5}{2}); (\frac{7}{2}, 4); (4, \frac{9}{2}); ...; (2n - \frac{1}{2}, 2n); (2n, 2n + \frac{1}{2}) \right).$$

4 The BRST symmetry and the consistent anomaly

The main advantage of this geometrical framework is to define the SW_n symmetries as gauge transformations of the vector bundle Φ and to provide a systematic method to derive a nilpotent BRST algebra, as we now discuss. This is a straightforward extension of the framework proposed by Zucchini [4] to formulate the symmetries of the induced light cone W_n-gravity. The most used path to study the quantum invariance of these theories consists in deriving from the underlying algebra the BRST charge Q. The knowledge of this operator is essential in a great body of work in W strings [30] towards unravelling the spectrum of physical states. The failure of Q^2 to vanish leads to an anomaly in the BRST operator algebra. However the obtention of anomalies in the BRST Ward identities requires the construction of a nilpotent BRST algebra. The two approaches are difficult to compare although in a recent paper [31] a relation between these two notions has been noticed.

4.1 BRST algebra

In sect.2 we have seen that the transition functions on overlapping domains of the local maps $\mathcal{Z}_\alpha^i \equiv (Z_\alpha^i, \Theta_\alpha^i)$ define an $sl(n + 1 \mid n)$ -valued 1-cocycle $\Phi_{\alpha\beta}$ on $S\Sigma$ which in turn corresponds to a flat $sl(n + 1 \mid n)$ vector bundle Φ on $S\Sigma$. Such bundle is canonically associated to a generalized projective structure and can be considered as a functional of the fields $(\tilde\rho_i, \tilde\mu_i)$. The variations of these fields which leave this bundle invariant are precisely the form of the super W_n-symmetry transformations. These variations are obtained from deformations of the maps $\mathcal{Z}^i$ which are defined by

$$\mathcal{Z}' = \frac{R \cdot \mathcal{Z}}{(R^{2n}, \mathcal{Z})_E}, \tag{35}$$

where R is an OSp$(n+1 \mid n)$ matrix, and R^{2n} is the $(2n+1)^{th}$ row (labelled $2n$) of R (R being written in diagonal grading). Requiring for consistency, that (Z, Θ) and (Z', Θ') glue on overlapping domains as in (2) i.e.

$$R_\alpha \Phi_{\alpha\beta} = \Phi_{\alpha\beta} R_\beta, \tag{36}$$

means that these coordinates are related by a gauge transformation of the flat vector bundle Φ defined by the matrix function R. The infinitesimal variations of the maps given in terms of infinitesimal parameters ϵ_j^i are

$$\delta \mathcal{Z}^i = \epsilon_k^i \mathcal{Z}^k - \epsilon_k^{2n} \mathcal{Z}^k \mathcal{Z}^i. \tag{37}$$

These transformations are generalization of the laws given by the group $\mathrm{OSp}(2\,|1)$ and which are the infinitesimal form of the well-known superconformal transformations [32] for the $N = 1$ SRS, namely

$$Z' = \frac{aZ+b}{cZ+d} + \Theta \frac{\alpha Z + \beta}{(cZ+d)^2}, \tag{38}$$

$$\Theta' = \frac{\alpha Z + \beta}{cZ+d} + \Theta \frac{1}{cZ+d}. \tag{39}$$

with R belonging to $\mathrm{OSp}(2|\,1)$, i.e.

$$R = \begin{pmatrix} a & \alpha b - \beta a & b \\ \alpha & 1 - \alpha\beta & \beta \\ c & \alpha d - \beta c & d \end{pmatrix}, \quad ad - bc = 1 + \alpha\beta.$$

Indeed, in infinitesimal form these transformations read

$$\delta Z = \epsilon_1 + 2\epsilon_0 Z - \epsilon_2 Z^2 - \epsilon_e \Theta - \epsilon_d \Theta Z, \tag{40}$$

$$\delta \Theta = \epsilon_d Z + \epsilon_e + \epsilon_0 \Theta - \epsilon_2 \Theta Z, \tag{41}$$

where the infinitesimal parameters ϵ_i, $i = 0, 1, 2$ and the ϵ_d, ϵ_e are Grassmann even and odd respectively. For the sake of comparison let us restrict ourselves to the Wess-Zumino gauge (W-Z) where $DZ = \Theta D\Theta$. If we introduce the infinitesimal parameter

$$\Upsilon = (\epsilon_1 + 2\epsilon_0 Z - \epsilon_2 Z^2 - 2\epsilon_e \Theta - 2\epsilon_d \Theta Z) \frac{1}{(D\Theta)^2}, \tag{42}$$

and assume for simplicity that the ϵ_i's are antiholomorphic, these variations become

$$\delta Z = \Upsilon(D\Theta)^2 - \Theta \delta\Theta, \tag{43}$$

$$\delta \Theta = \frac{1}{2} D\Upsilon D\Theta + \Upsilon \partial\Theta. \tag{44}$$

They are identical to the BRST laws given in [12] which read

$$sZ = C^z(D\Theta)^2 - \Theta s\Theta, \tag{45}$$

$$s\Theta = \frac{1}{2}DC^zD\Theta + C^z\partial\Theta, \tag{46}$$

when the infinitesimal parameter Υ has been turned into the ghost field C^z and the gauge transformation δ into the BRST operator s. These laws give the forms of the BRST transformations of the superprojective connection and of the super-Beltrami differential, which are the current and the gauge superfield respectively of the induced supergravity in the W-Z gauge. Thus the well-known results of the $N = 1$ SRS [12] restricted to this gauge are easily reproduced.

Now from the construction of the generalized superprojective connections and super-Beltrami differentials given in sect.2, and thanks to the transformations (37) we generalize the above result to the formulation of the nilpotent BRST algebra corresponding to the classical SW_n symmetry.

The super determinant Δ' of W_0' obtained by replacing in (3) the $\mathcal{Z}$'s by the $\mathcal{Z}''$'s defined by eq.(35) is

$$\Delta' = \frac{\Delta}{(R^{2n}, \mathcal{Z})_E} sdet\Big([\kappa^{(l)}\tilde{W}_0 D^{(2l+1)}(R^{st}) + \kappa^{(l)'}W_0 D^{(2l)}(R^{st})]W_0^{-1}\Big). \tag{47}$$

where $\kappa^{(l)}$ and $\kappa^{(l)'}$ denote numerical matrices which are respectively defined by

$$\kappa^{(l)}_{2q+1,p} = C^l_{n-1-q}\delta_{p,2(q+l+1)}; \qquad \kappa^{(l)}_{2q,p} = 0.$$

$$\kappa^{(l)'}_{2q+1,p} = C^l_{n-1-q}\delta_{p,2(q+l)+1}; \qquad \kappa^{(l)'}_{2q,p} = C^l_{n+q}\delta_{p,2(q+l)}.$$

It then follows that

$$[\kappa^{(l)}, \kappa^{(m)}] = [\kappa^{(l)}, \kappa^{(m)'}] = [\kappa^{(l)'}, \kappa^{(m)'}] = 0. \tag{48}$$

From the definition of W in terms of W_0 (given by eqs.(3, 4)) it is easy to show that

$$W = \Xi W_0, \tag{49}$$

where the matrix Ξ can be written in terms of κ and κ'

$$\Xi_{q,2l} = \sum_{p=0}^{2n} \kappa^{(n-l)'}_{q,2n-p} D^{(p)}(\Delta^{-1}),$$

$$\Xi_{q,2l+1} = \sum_{p=0}^{2n} \kappa^{(n-1-l)}_{q,2n-p} D^{(p)}(\Delta^{-1}).$$

We can verify that

$$\Xi\kappa^{(l)'} = \kappa^{(l)'}\Xi \quad \text{and} \quad \Xi\kappa^{(l)} = \kappa^{(l)}\tilde{\Xi}.. \tag{50}$$

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By using these relations, one can replace W_0 and $\tilde{W}_0$ by W and $\tilde{W}$ respectively in (47)

$$\Delta' = \frac{\Delta}{(R^{2n}, \mathcal{Z})_E} sdet\Big([\kappa^{(l)}\tilde{W}D^{(2l+1)}(R^{st}) + \kappa^{(l)\prime}WD^{(2l)}(R^{st})]W^{-1}\Big). \qquad (51)$$

In order to obtain the BRST algebra corresponding to (51), let us consider the infinitesimal parametrization of these transformations by setting $R = 1 + \epsilon$ (with str(ϵ)=0). This linearizes the r.h.s of (51) which becomes

$$\delta\Delta = -\Delta\Big[(\epsilon^{2n}, \mathcal{Z})_E - str\Big((\sum_{l=0}^{n-1} \kappa^{(l)}\tilde{W}D^{(2l+1)}(\epsilon^{st}) + \sum_{l=0}^{n} \kappa^{(l)\prime}WD^{(2l)}(\epsilon^{st}))W^{-1}\Big)\Big],$$
$$(52)$$

where ϵ^{2n} is the $(2n+1)^{th}$ row of ϵ. When the infinitesimal variations (37) of the maps are written in terms of a ghost matrix superfield γ (in diagonal basis) instead of infinitesimal parameters they become the BRST transformations corresponding to the classical super W-symmetries. The matrix elements γ_s^r, such that $r + s$ is even are nilpotent (i.e.$(\gamma_s^r)^2 = 0$) whereas the remaining entries have both a ghost number one and a grassmannian character. The BRST laws obeyed by the maps $\mathcal{Z}^i$ are [11]

$$s\mathcal{Z}^i = \gamma_k^i \mathcal{Z}^k - \gamma_k^{2n} \mathcal{Z}^k \mathcal{Z}^i. \qquad (53)$$

Nilpotency of the law (53) is fulfilled when

$$s\gamma = -\gamma^2. \qquad (54)$$

Therefore this analysis allows to construct with the help of γ and W a matrix C with ghost grading one

$$C = \Big(\sum_{l=0}^{n-1} \kappa^{(l)}\tilde{W}D^{(2l+1)}(\frac{\hat{\gamma}^{st}}{\Delta}) + \sum_{l=0}^{n} \kappa^{(l)\prime}WD^{(2l)}(\frac{\hat{\gamma}^{st}}{\Delta})\Big)W^{-1}, \qquad (55)$$

where

$$\hat{\gamma} = \gamma - str\Big(\Big(\sum_{l=0}^{n-1}[\kappa^{(l)}\tilde{W}D^{(2l+1)}(\gamma^{st})] + \sum_{l=0}^{n}[\kappa^{(l)\prime}WD^{(2l)}(\gamma^{st})]\Big)W^{-1}\Big)\mathbf{1}. \qquad (56)$$

From (52) and (54) it follows that

$$s(\mathcal{Z}^r\Delta^{-1}) = \hat{\gamma}_j^r \mathcal{Z}^j\Delta^{-1}. \qquad (57)$$

---

[11] As usual the operator s acts as an antiderivation from the right, the grading being defined by the sum of the ghost number and the form degree; s does not feel the Grassmann parity.

From (57) it is then straightforward to deduce the BRST transformation of the matrix W

$$sW = CW. \tag{58}$$

By construction this super-matrix C, which is traceless (as it is straightforward to verify from (55,56)), is independent from map choices in the super-projective structure $(Z^i_\alpha, \Theta^i_\alpha)$. Moreover, under a superconformal coordinate change in the superholomorphic canonical bundle (given by the transition matrix Λ) this superfield transforms as

$$C_b = \Lambda_{ba} C_a \Lambda_{ba}^{-1}. \tag{59}$$

The overall consistency of this framework is given by

$$sC = -CC. \tag{60}$$

This law can be proved starting from the expression (55) of C : using (56) C can be written

$$C = B - \sum_{l=0}^{n-1} \kappa^{(l)} D^{2l+1} (\frac{\gamma_D^{st}}{\Delta}) - \sum_{l=0}^{n} \kappa^{(l)\prime} D^{2l} (\frac{\gamma_D^{st}}{\Delta}),$$

with

$$B = \left(\sum_{l=0}^{n-1} \kappa^{(l)} \tilde{W} D^{(2l+1)} (\frac{\gamma^{st}}{\Delta}) + \sum_{l=0}^{n} \kappa^{(l)\prime} W D^{(2l)} (\frac{\gamma^{st}}{\Delta}) \right) W^{-1}.$$

and

$$\gamma_D = (str B) \mathbf{1}.$$

By direct application of s on B with the help of the explicit expressions of $\kappa^{(l)}$ and $\kappa^{(l)\prime}$ it is possible to show that

$$sB = -C^2.$$

From the tracelessness property of C^2 (which is a result of $str C = 0$ and of the ghost grading one of every issue of C) it follows that $s\gamma_D = 0$ and consequently $sC = -C^2$.

From (58) we readily derive the BRST transformation law for Ω and Ω^*

$$s\Omega = DC + \tilde{C}\Omega - \Omega C \tag{61}$$

$$s\Omega^* = \bar{D}C + \tilde{C}\Omega^* - \Omega^* C. \tag{62}$$

Now we explain in the following how this matrix formalism allows us to find the particular BRST algebra which is obeyed by the fields of a given SW_n model. It is well known that the Ward identities for the induced W-gravity are very similar in structure to the BRST transformations of the projective connection. On the usual Riemann surface this relation is a straightforward consequence of the striking similarity between the Beltrami equation and the BRST transformation

of the projective coordinate Z. The same sort of relations, discussed previously for W_n models in ref.[18], are also present in the SW_n models. Indeed the comparison of (13) and (61) shows that the replacement of $(\bar{D}, \Omega^*)$ by (s, C) in (13) leads to (61) up to some signs [12]. This allows us to derive the explicit form of C from Ω^* by replacing $\mu_{\bar{\theta}}^{\theta\cdots}$ by $c^{\,\theta\cdots}$, with the substitution of a ghost degree to the conformal index $\bar{\theta}$.

From the relations (61,62) we compute the BRST laws of the superfields ρ_i, of the generalized Beltrami coefficients μ_i and of the superghosts c_i. From (59) and the transition laws (11,12) it can be checked easily that the BRST laws (60,61,62) are invariant under a superconformal coordinate change. Thus they are well defined on the SRS.

In summary the laws (60,61,62) represent the nilpotent BRST algebra (as it can be verified by an explicit calculation) corresponding to a given classical SW-algebra. They are obtained thanks to the definition (55) which induces the BRST transformations (58,60), once the law (54) has been chosen.

4.2 Super covariant anomalies

Using the coboundary operator d defined as

$$d\Phi = D\Phi dz + \bar{D}\Phi d\bar{z}$$

and the laws (61,62) we can write the transformation of the connection $\mathcal{A}$ defined in (32) :

$$
\begin{aligned}
s\mathcal{A} &= -dC - \tilde{C}\mathcal{A} - \mathcal{A}\tilde{C} & (63) \\
s\tilde{\mathcal{A}} &= -d\tilde{C} - C\tilde{\mathcal{A}} - \tilde{\mathcal{A}}C & (64)
\end{aligned}
$$

where $\widetilde{dC} = d\tilde{C}$ results from a cancellation of minus signs between the derivative $\widetilde{DC} = -D\tilde{C}$ and $\tilde{dz} = -dz$. The polynomial T_3^0 of rank 3 (where the lower index denotes the form degree and the upper index the ghost number)

$$T_3^0 = str(\mathcal{A}D\mathcal{A} + \frac{2}{3}\mathcal{A}\mathcal{A}\mathcal{A}).$$

generates a tower of descent equations through the application of the BRST transformations (60,63,64)

$$
\begin{aligned}
sT_3^0 + dT_2^1 &= 0 \\
sT_2^1 + dT_1^2 &= 0 & (65) \\
sT_1^2 + dT_0^3 &= 0 & (66) \\
sT_0^3 &= 0, & (67)
\end{aligned}
$$

---

[12]This sign difference results from the fact that the operator D acts from the left, whereas s acts from the right.

where the explicit expressions of the cocycles are given by

$$
\begin{align}
T_2^1 &= -str(\tilde{C}\Omega\tilde{\Omega}^* + \tilde{C}\Omega^*\tilde{\Omega})dzd\bar{z} \tag{68}\\
T_1^2 &= str(C^2\tilde{\Omega})dz + str(C^2\tilde{\Omega}^*)d\bar{z} \tag{69}\\
T_0^3 &= -\tfrac{1}{3}str(\tilde{C}^3). \tag{70}
\end{align}
$$

Eq.(65) implies $s \int T_2^1 = 0$ and identifies this descendant as a candidate for a consistent anomaly, a non-trivial solution of the Wess-Zumino [33] consistency condition.

The anomalous cocycle T_2^1 does not transform tensorially, as can be easily verified by using eqns. (11,12,59). However it can be written, using (13),

$$
T_2^1 = str(\tilde{C}\bar{D}\tilde{\Omega} + \tilde{C}D\tilde{\Omega}^*)dzd\bar{z}.
$$

As one can straightforwardly verify, only the first term of the sum

$$
\sigma_2^1 = str(\tilde{C}\bar{D}\tilde{\Omega}) \tag{71}
$$

is well-defined on a SRS. It also solves the descent equations (65-67) and is thus a candidate for a covariant and consistent anomaly [13]. The corresponding cocycles are

$$
\begin{align}
\sigma_1^2 &= \tfrac{1}{2}str(CD\tilde{C} + 2C\tilde{\Omega}\tilde{C})dz + \tfrac{1}{2}str(\bar{D}\tilde{C}\tilde{C})d\bar{z} \tag{72}\\
\sigma_0^3 &= \tfrac{1}{6}str(\tilde{C}^3). \tag{73}
\end{align}
$$

The fact that $\mathcal{A}$ is not a generic connection since the matrix elements of Ω and Ω^* are not all independent but linked by eq.(13) plays no role here (C is not constrained since, as mentionned before, (13) becomes (61), the transformation law of Ω, after suitable substitutions). Actually, the nilpotency of s, which is a crucial ingredient in this framework, is independent of these constraints, as it can be straightforwardly ascertained from (60,61,62). At last the residual conditions given by (13) after the determination of some entries of Ω^*, are the super holomorphy conditions obeyed by the $\tilde{\rho}_i$; they serve, as explained above, to relate the two sets (68,69,70) and (71,72,73) of cocycles .

The formalism presented above provides us with a completely algorithmic procedure of calculating the anomaly associated to a given super W-model and the cocycles related to this anomaly by the system of descent equations. Moreover the solution (71,72,73) has the advantage of being defined on a generic SRS of arbitrary genus. The form of the Virasoro Ward identity on an arbitrary Riemann surface was first derived in [35]; however there an holomorphic projective

---

[13]Covariant expressions for the anomaly have already been obtained in [26] and for the bosonic case in [4, 18, 34].

connection $\mathcal{R}$, which was BRST inert, was introduced by hand. In contrast, the formulation presented here has the advantage of being self-contained since it is the usual projective connection that renders the local expression of the super-anomaly well-defined.

Since we do not start as usual from an action, but construct in an algebraic way expressions of the BRST anomaly and of its cocycles which obey to the descent equations, let us discuss more precisely the kind of anomalies we obtain. The so-called universal W-gravity anomalies (for a review see ref.[2]) that are present in all theories of matter coupled to W-gravity, are those anomalies that depend only on the gauge fields $\tilde{\mu}_i$ and not on the matter fields. For theories in which the symmetry is non-linearly realized the universal form of the anomaly in the spin s symmetry (by reference to the spin of the corresponding current $\tilde{\rho}_i$) is given by $\tilde{\mu}_i \partial^{s+1} c^i$ where c^i denotes the ghost associated to this symmetry. In a framework where a special realization of the currents in terms of scalar fields is considered, these anomalies arise at $s - 1$ loops level. At lower number of loops there are anomalies which depend on matter fields. The supersymmetric extension of these universal anomalies is the subject of this chapter, where the equivalent of the corresponding universal expression given above is dressed with $\tilde{\rho}_i$ dependent terms in order to insure the BRST invariance for consistent anomalies and both BRST invariance and conformally covariance for covariant anomalies.

5 Supergravity

This section is an illustration of the general formulation presented here for the particular case $n = 1$ (i.e. 3×3 matrices). This corresponds to the SRS approach of the $(1, 1)$ supersymmetry. First, the model is studied in its full generality and then the restricted geometry given by the W-Z gauge is discussed thus making contact with the usual supergravity and the super-Virasoro algebra . Finally advantages with respect to previous approaches are emphasized.

5.1 The underlying classical super-Virasoro algebra

We begin with a connection form $\mathcal{A}$ built from the matrix W through the definitions (9,10). The two components of this connection 1-form are, in the block grading,

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ \rho_3 & 0 & \rho_2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Omega^* = \begin{pmatrix} \alpha_1 & \mu_3 & a_1 \\ \alpha_2 & \alpha_3 & a_2 \\ a_3 & \mu_2 & \alpha_4 \end{pmatrix},$$

and equivalently in the diagonal grading

$$\Omega = \begin{pmatrix} 0 & \rho_2 & \rho_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Omega^* = \begin{pmatrix} \alpha_3 & a_2 & \alpha_2 \\ \mu_2 & \alpha_4 & a_3 \\ \mu_3 & a_1 & \alpha_1 \end{pmatrix}.$$

The conformal indices of the four independent fields are

$$\rho_2 \mid_z \quad \rho_3 \mid_{z\theta} \quad \mu_2 \mid_\theta^\theta \quad \mu_3 \mid_\theta^z .$$

Building explicitly the matrix Λ and using relations (11) and (12), we determine the variation of the coefficients under a superconformal change of coordinates $z \longrightarrow z'$ (we recall that $e^{-X} = D\theta'$)

$$\rho_3' = e^{3X} [\rho_3 + S(z,\theta;\theta') + (DX)\rho_2] \tag{74}$$
$$\rho_2' = e^{2X} \rho_2 \tag{75}$$

$$\mu_3' = e^{\bar{X}} e^{-2X} \mu_3 \tag{76}$$
$$\mu_2' = e^{\bar{X}} e^{-X} (\mu_2 + (DX)\mu_3) \tag{77}$$

where $S(z,\theta;\theta')$ is the super-Schwarzian derivative of this superconformal change of coordinates [36]. Then if $\rho_2 = 0$, ρ_3 transforms as a superprojective connection.

The matrix W considered as an element of $sl(2 \mid 1)$ can be parametrized in the diagonal basis using a Gauss decomposition

$$W = \begin{pmatrix} 1 & 0 & 0 \\ \phi_2 & 1 & 0 \\ f & \phi_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1}\lambda_2^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 & \Phi_1 & F \\ 0 & 1 & \Phi_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{78}$$

where the elements Φ_i, and ϕ_i are Grassmann variables. Comparing with (4) we can express all the variables in terms of the superprojective coordinates Z and Θ. They are given by:

$$\phi_1 = Dln\lambda_1, \qquad \lambda_1 = (D\Theta), \qquad \Phi_1 = \Theta,$$

$$\phi_2 = -Dln\Delta, \qquad \lambda_2 = D(\tfrac{DZ}{D\Theta})(D\Theta)^{-1}, \qquad \Phi_2 = (DZ)\lambda_1^{-1},$$

$$f = -\partial ln\Delta, \qquad\qquad\qquad F = Z,$$

where Δ is

$$\Delta = D(\frac{DZ}{D\Theta}). \tag{79}$$

The entries of the Ω matrix α_i and a_i are determined by solving the system given by (13) when $\bar{D}\Omega$ has a null contribution. The remaining equations of this system are the holomorphy conditions for ρ_3 and ρ_2. They take the forms

$$\begin{aligned}
\bar{D}\rho_3 - \mu_3\partial\rho_3 - \mu_2 D\rho_3 - (2\partial\mu_3 + D\mu_2)\rho_3 + \partial^2\mu_2 - \rho_2\partial\mu_2 &= 0, \\
\bar{D}\rho_2 - \partial^2\mu_3 - 2D\partial\mu_2 - \partial(\mu_3\rho_2) + \rho_3 D\mu_3 - \mu_2 D\rho_2 + 2\rho_3\mu_2 &= 0.
\end{aligned} \tag{80}$$

To exploit these relations we have to define the physical fields and first to give a physical meaning to ρ_3, i.e. to turn this field into a superprojective connection.

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From the expressions (74) and (75) it is easy to guess the right term to add to ρ_3. Of course the Drinfeld-Sokolov method described in sect. 3.1 leads to the same result which is

$$\begin{aligned} \tilde{\rho}_2 &= \tfrac{1}{2}\rho_2, \\ \tilde{\rho}_3 &= \rho_3 - \tfrac{1}{2}D\rho_2. \end{aligned} \tag{81}$$

To covariantize μ_3 we use the method of the supersymplectic form (see sect. 3.2) by considering :

$$\omega = \int_\Sigma (\delta\mu_3 \wedge \delta\rho_3 + \delta\mu_2 \wedge \delta\rho_2),$$

and replacing the ρ_k by the $\tilde{\rho}_k$:

$$\begin{aligned} \omega &= \int_\Sigma (\delta\mu_3 \wedge \delta\tilde{\rho}_3 + \delta\mu_3 \wedge \delta D\tilde{\rho}_2 + \delta\mu_2 \wedge \delta(2\tilde{\rho}_2)) \\ &= \int_\Sigma (\delta\mu_3 \wedge \delta\tilde{\rho}_3 + \delta(2\mu_2 + D\mu_3) \wedge \delta\tilde{\rho}_2). \end{aligned}$$

We thus obtain :

$$\begin{aligned} \tilde{\mu}_3 &= \mu_3 \\ \tilde{\mu}_2 &= 2\mu_2 + D\mu_3 \end{aligned} \tag{82}$$

and as one can directly verify these new fields transform tensorially.

Let us note that the Gauss decomposition induces for these fields surprisingly simple expressions

$$\tilde{\rho}_2 = -\frac{1}{2}\partial ln\lambda_2 + \frac{1}{2}Dln\lambda_1 Dln\lambda_2, \tag{83}$$

$$\tilde{\rho}_3 = -S(z,\theta;\Theta) - \frac{1}{2}\partial Dln\lambda_2 + \frac{1}{2}Dln\lambda_1\partial ln\lambda_2 + \frac{1}{2}Dln\lambda_2\partial ln\lambda_1. \tag{84}$$

These expressions involve only the parameters entering in the central matrix of the Gauss decomposition (78). On an ordinary SRS, from the $\frac{1}{2}$-super differential λ_1 is built the super-affine connection $\zeta = Dln\lambda_1$ which, in turn allows one to define the super-Schwarzian derivative :

$$S(z,\theta;\Theta) = \partial\zeta - \zeta D\zeta. \tag{85}$$

In this kind of generalization of the SRS, the partner λ_2 of λ_1 appears in (83,84) in expressions very reminiscent of the usual super-affine connection (85). However the terms in the right hand side of (84) (except S of course) and $\tilde{\rho}_2$ transform covariantly.

Now we replace the ρ_k's and the μ_k's by the corresponding $\tilde{\rho}_k$'s and $\tilde{\mu}_k$'s in the identities (80) and obtain

$$\bar{D}\tilde{\rho}_3 = \frac{1}{2}\mathcal{L}_2(-\tilde{\rho}_3)\tilde{\mu}_3 + (\tilde{\rho}_2\partial - \frac{1}{2}D\tilde{\rho}_2D + \frac{1}{2}\partial\tilde{\rho}_2)\tilde{\mu}_2, \tag{86}$$

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$$\bar{D}\tilde{\rho}_2 = \frac{1}{2}\mathcal{L}_1(-\tilde{\rho}_3)\tilde{\mu}_2 + (\tilde{\rho}_2\partial - \frac{1}{2}D\tilde{\rho}_2 D + \partial\tilde{\rho}_2)\tilde{\mu}_3, \tag{87}$$

where $\mathcal{L}_1(-\tilde{\rho}_3)$ and $\mathcal{L}_2(-\tilde{\rho}_3)$ are the super Bol operators [26] of the superprojective connection $\mathcal{R} = -\tilde{\rho}_3$

$$\begin{aligned}
\mathcal{L}_1(\mathcal{R}) &= D^3 + \mathcal{R}, \\
\mathcal{L}_2(\mathcal{R}) &= D^5 + 3\mathcal{R}D^2 + (D\mathcal{R})D + 2(D^2\mathcal{R}).
\end{aligned}$$

The algebraic content of these holomorphy equations is given by the Poisson brackets among the spin-one superfield $\tilde{\rho}_2$ and the superfield of spin $3/2$ $\tilde{\rho}_3$. It corresponds to the $N = 2$ classical super Virasoro algebra [37]. If we further impose that the connection $1-$form be an $osp(1\,|\,2)$ connection instead of an $sl(2|1)$ one, the fields $\tilde{\rho}_2$ and $\tilde{\mu}_2$ become zero. By an explicit calculation of ρ_2 in terms of the coordinates Z and Θ we found that setting $DZ = \Theta D\Theta$ brings ρ_2 to zero and ρ_3 to $-S$.

5.2   BRST analysis and the covariant anomaly

The BRST transformations of the superprojective coordinates are

$$\begin{aligned}
sZ &= \tilde{c}_3\partial Z + \frac{1}{2}D\tilde{c}_3 DZ - \frac{1}{2}\tilde{c}_2 DZ \tag{88} \\
s\Theta &= \frac{1}{2}D\tilde{c}_3 D\Theta + \tilde{c}_3\partial\Theta - \frac{1}{2}\tilde{c}_2 D\Theta \tag{89}
\end{aligned}$$

They are given in terms of the covariant (tilde) fields $\tilde{c}_k$ whose expressions as functions of the c_i are similar to (82).

$$\tilde{c}_3 = c_3, \qquad \tilde{c}_2 = 2c_2 - Dc_3. \tag{90}$$

Then the construction of the BRST algebra follows from (60-62) which insure its completeness and its nilpotency. It is given here in terms of the covariant (tilde) fields $\tilde{\rho}_k$, $\tilde{\mu}_k$, defined by (81,82). We obtain

$$\begin{aligned}
s\tilde{c}_3 &= -\tilde{c}_3\partial\tilde{c}_3 - \frac{1}{4}D\tilde{c}_3 D\tilde{c}_3 + \frac{1}{4}\tilde{c}_2\tilde{c}_2, \tag{91} \\
s\tilde{c}_2 &= -\tilde{c}_3\partial\tilde{c}_2 - \frac{1}{2}\tilde{c}_2\partial\tilde{c}_3 - \frac{1}{2}D\tilde{c}_3 D\tilde{c}_2. \tag{92}
\end{aligned}$$

The fields $\tilde{\rho}_2$, $\tilde{\rho}_3$ obey the following laws

$$\begin{aligned}
s\tilde{\rho}_3 &= -\frac{1}{2}\mathcal{L}_2(-\tilde{\rho}_3)\tilde{c}_3 + (\tilde{\rho}_2\partial - \frac{1}{2}D\tilde{\rho}_2 D + \frac{1}{2}\partial\tilde{\rho}_2)\tilde{c}_2, \tag{93} \\
s\tilde{\rho}_2 &= -\frac{1}{2}\mathcal{L}_1(-\tilde{\rho}_3)\tilde{c}_2 + (\tilde{\rho}_2\partial - \frac{1}{2}D\tilde{\rho}_2 D + \partial\tilde{\rho}_2)\tilde{c}_3, \tag{94}
\end{aligned}$$

and the $\tilde{\mu}_2$, $\tilde{\mu}_3$ fields satisfy

$$
\begin{aligned}
s\tilde{\mu}_3 &= \bar{D}\tilde{c}_3 + \tilde{c}_3\partial\tilde{\mu}_3 - \tilde{\mu}_3\partial\tilde{c}_3 + \frac{1}{2}D\tilde{\mu}_3 D\tilde{c}_3 + \frac{1}{2}\tilde{c}_2\tilde{\mu}_2, \quad\quad (95)
\end{aligned}
$$

$$
\begin{aligned}
s\tilde{\mu}_2 &= \bar{D}\tilde{c}_2 - \frac{1}{2}\tilde{\mu}_2\partial\tilde{c}_3 + \frac{1}{2}D\tilde{c}_3 D\tilde{\mu}_2 + \tilde{c}_3\partial\tilde{\mu}_2 - \frac{1}{2}\tilde{c}_2\partial\tilde{\mu}_3 \\
&\quad + \frac{1}{2}D\tilde{\mu}_3 D\tilde{c}_2 - \tilde{\mu}_3\partial\tilde{c}_2. \quad\quad (96)
\end{aligned}
$$

Having at hand the BRST relations for all the fields, we can address the problem of the anomaly. Using relation (71) we can write down explicitly

$$
\begin{aligned}
\sigma_2^1 &= \{\tilde{c}_2(-\mathcal{L}_1(-\tilde{\rho}_3)\tilde{\mu}_2 + D\tilde{\mu}_3 D\tilde{\rho}_2 - 2\tilde{\rho}_2 D^2\tilde{\mu}_3 - 2\tilde{\mu}_3 D^2\tilde{\rho}_2) + \\
&\quad \tilde{c}_3(\mathcal{L}_2(-\tilde{\rho}_3)\tilde{\mu}_3 - D\tilde{\rho}_2 D\tilde{\mu}_2 + \tilde{\mu}_2 D^2\tilde{\rho}_2 + 2\tilde{\rho}_2 D^2\tilde{\mu}_2)\}d\boldsymbol{z}d\bar{\boldsymbol{z}}. \quad\quad (97)
\end{aligned}
$$

It is worth noting that by symmetrizing this covariant anomaly we get the compact form $\sigma_2^1 = \sum_{i=2}^{3}(\tilde{\mu}_i s\tilde{\rho}_i - \tilde{c}_i\bar{D}\tilde{\rho}_i)d\boldsymbol{z}d\bar{\boldsymbol{z}}$.

5.3   Comparison with previous works

Super-Beltrami differentials were previously introduced either by using zweibeins [38] or with the help of super 1-forms [12]. In this latter approach super-Beltrami differentials occur without any reference to metrics or vielbeins thanks to the super 1-forms $e^Z \equiv dZ + \Theta d\Theta$ and $e^\Theta \equiv d\Theta$ (and c.c.) which span the cotangent space of the SRS. Their expressions with respect to a reference coordinate system yields six superfields $(H_{\bar{z}}{}^z, H_\theta{}^z, H_{\bar{\theta}}{}^z, H_\theta{}^\theta, H_{\bar{z}}{}^\theta, H_{\bar{\theta}}{}^\theta)$. The structure equations $de^Z + \Theta d\Theta = 0 = de^\Theta$ (and c.c.) relate four of these superfields to only two independent Beltrami coefficients $H_\theta{}^z, H_{\bar{\theta}}{}^z$. In our parametrization we also obtain two independent fields $\tilde{\mu}_2$ and $\tilde{\mu}_3$, which could be compared, due to their conformal weights, to $H_{\bar{\theta}}{}^z$ and $H_{\bar{\theta}}{}^\theta$, namely

$$
H_{\bar{\theta}}{}^z = \tilde{\mu}_3 + (DZ - \Theta D\Theta)\frac{D\Theta\partial\Theta\bar{D}Z}{(\partial Z)^2 D\Theta}
$$

$$
H_{\bar{\theta}}{}^\theta = -\frac{1}{2}\frac{D\Theta}{\sqrt{\partial Z + \Theta\partial\Theta}}(\tilde{\mu}_2 + D\tilde{\mu}_3) - (DZ - \Theta D\Theta)\frac{\bar{D}\Theta\partial\Theta}{(\partial Z)^{3/2}D\Theta}.
$$

Whereas our fields transform homogeneously under a superconformal change of the coordinate system (more precisely they are sections $e^{\bar{X}}e^{-X}$ and $e^{\bar{X}}e^{-2X}$, respectively, of the canonical fibre bundle), the transformation laws of $H_{\bar{\theta}}{}^z$ and $H_{\bar{\theta}}{}^\theta$ do not take simple forms. They depend in particular on the superfield $H_\theta{}^z$. Furthermore, our fields $\tilde{\mu}_2$ and $\tilde{\mu}_3$ appear in the context of $sl(2\,|\,1)$, while, in contrast, it is possible to obtain both $H_\theta{}^z$ and $H_{\bar{\theta}}{}^z$ in the context of $osp(1\,|\,2)$ [39].

The choice made in sect.5.2 [14] for the two independent superghosts c_3 and c_2 is not unique. Indeed, if we take as generators of the BRST transformations the superghost fields c^z, c^θ defined by

$$
\begin{aligned}
C^Z &= c^z \Lambda_z{}^Z \\
C^\Theta &= c^\theta \sqrt{\Lambda_z{}^Z} + c^z \partial\Theta
\end{aligned}
$$

where $\Lambda_z{}^Z = \partial Z + \Theta\partial\Theta$ and where C^Z and C^Θ are

$$
\begin{aligned}
C^Z &= \gamma_1 + 2\gamma_0 Z - \gamma_2 Z^2 \\
C^\Theta &= \gamma_d Z + \gamma_e + \gamma_0 \Theta - \gamma_2 \Theta Z
\end{aligned}
$$

we obtain instead of (88,89) the BRST laws given in [12]. This remark completes the comparison between our formalism and this work; both approaches start from the same gauge transformations, namely the relations (40,41) but then differ by the choice of the infinitesimal parameters which turned into ghost fields become the generators of the BRST transformations.

The well-known results of the $N = 1$ SRS [12, 13] restricted to the W-Z gauge, where $H_{\bar\theta}{}^z = 0$ ($\tilde\mu_2 = 0$), are easily reproduced. In this case the pair of geometrical superfields usually encountered in the literature, namely the super Schwarzian derivative $S(\Theta; z, \bar z, \theta, \bar\theta)$ [36] and the super Beltrami superfield $H_{\bar\theta}{}^z$ [12] correspond respectively to $\tilde\rho_3$ and $\tilde\mu_3$. Moreover, the holomorphy condition for $\tilde\rho_3$, which is the superconformal anomalous Ward identity obtained by replacing $\tilde\rho_3 \to \frac{\delta\Gamma}{\delta\mu}$ (where Γ is the generating functional for current correlation functions), is recovered and appears, as expected, as a compatibility condition between $\tilde\rho_3$ and $\tilde\mu_3$ following from (86) by putting $\tilde\mu_2 = 0$. Otherwise the superghost C^z is given by $\tilde c_3$, the laws (91,93,95) reduce to the well known transformations [8, 40] and it is not very hard to check that (97) gives the standard super-diffeomorphism anomaly [8, 12].

We finally briefly discuss the SW_3 example ($n = 2$) by comparing it with existing results [9]. Previous studies of this case are few and uncomplete. However in [9] the link between the $N = 1$ super W-algebra and $osp(2m \pm 1|2m)$ was

---

[14] We can use the formula (55) to express the superghosts $\tilde c_3$, $\tilde c_2$ which are the entries $<20>$ and $<10>$ respectively of the matrix C in terms of the superghosts γ

$$
\begin{aligned}
\tilde c_3 &= (\gamma_1 + 2\gamma_0 Z - \gamma_2 Z^2 - \gamma_e \Theta - \gamma_d \Theta Z)\frac{1}{\partial Z}(1 - \frac{DZ\partial\Theta}{\partial Z D\Theta}) \\
&- (\gamma_d Z + \gamma_e + \gamma_0 \Theta - \gamma_2 \Theta Z)\frac{DZ}{\partial Z D\Theta}, \\
\tilde c_2 &= -D c_3 - (\gamma_d Z + \gamma_e + \gamma_0 \Theta - \gamma_2 \Theta Z)\frac{1}{D\Theta}(1 + \frac{DZ\partial\Theta}{\partial Z D\Theta}) \\
&+ (\gamma_1 + 2\gamma_0 Z - \gamma_2 Z^2 - \gamma_e \Theta - \gamma_d \Theta Z)\frac{\partial\Theta}{\partial Z D\Theta},
\end{aligned}
$$

where we have assumed for simplicity that the γ's are antiholomorphic.

studied. The authors developed the example of $osp(3|2)$ and gave the Poisson brackets among fields V_i's which are related to our fields $\tilde{\rho}_i$'s through the correspondence $V_i \longleftrightarrow -\tilde{\rho}_i$, $3H_i \longleftrightarrow \tilde{\mu}_i$. A more general treatment of this example is given elsewhere [41].

6 Discussion and outlook

Let us conclude on some future prospects. Amazingly the theory presented here appears to yield new bosonic models. In the conventional geometrical framework [4, 8], the spins of the generators of the algebra are limited to $(2, 3, \ldots, n)$. In the present formalism the spin content of the supergenerators is given by $((1, \frac{3}{2}), (\frac{3}{2}, 2), (2, \frac{5}{2}) \ldots, (\frac{2n-1}{2}, n))$ where the decomposition in components has been made explicit. The limit of purely bosonic generators $(1, 2, 2, 3, \ldots, n)$ which is obtained when the expansion in component fields is limited to the scalar term for even spin generators and to the θ term for odd spin generators, supplies a new spin 1 current and duplicates the standard currents. Thus such a framework interestingly enough allows one to construct a spin 1 field which cannot be obtained in the conventional bosonic approach. Hence, for the first time, to our knowledge, a bosonic limit of a supersymmetric framework is obtained which is impossible to get in the standard bosonic scheme.

Moreover it is alluring to study these supersymmetric models since, compared to the underlying bosonic theories, they contain a rich gauge choice, generalizing the W-Z gauge. Particular subsets of fields are selected by setting to zero some superfields $\tilde{\rho}_i$. Then the holomorphy relations obeyed by these superfields become constraints for the super-Beltrami differentials. These constraints are more or less tractable. Two situations can occur.

Thanks to the constraints it might be straightforward to eliminate some of the super-Beltrami differentials as explicit functions of the remaining ones. In general these relations can be associated to some group prescription by assuming that the matrix Ω belongs to some representation of the Lie algebra of a super group. This is the case for instance for the choice made in [9] for SW_3.

More difficult is the situation where the constraints appear as differential equations implying an implicit dependence of some Beltrami differentials on other ones. In that case it seems always possible to extract the classical super algebras corresponding to this choice, without being able to write the corresponding Ward identities and anomaly. This indicates that these algebras cannot be used in a construction of some supersymmetric generalization of string theory and thus, gives a criterion to determine if the classical super algebra has a physical meaning or not.

Acknowledgments

7 Appendix

For $sl(n+1|n)$ we take the following Cartan matrix [22]

$$C_{i,j} = (-1)^{i+1}\delta_{i+1,j} + (-1)^i\delta_{i,j+1}.$$

The Chevalley basis in the diagonal grading is given by

$$
\begin{aligned}
h_i &= (-1)^{i+1}(E_{i,i} + E_{i+1,i+1}) \\
e_i &= (-1)^{i+1}E_{i,i+1} \\
f_i &= E_{i+1,i}
\end{aligned}
$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We have the relations

$$
\begin{aligned}
[h_i, e_j] &= C_{ij}e_j \\
[h_i, f_j] &= -C_{ij}f_j \\
[e_i, f_j] &= \delta_{ij}h_i.
\end{aligned}
$$

To construct the $osp(1|2)$ subalgebra generating matrices we need the inverse of the Cartan matrix. It is given by

$$
\begin{aligned}
C_{2p,j}^{-1} &= \sum_{k=0}^{p-1}\delta_{2k+1,j} \quad 1 \le p \le n \\
C_{2p+1,j}^{-1} &= \sum_{k=p+1}^{n}\delta_{2k,j} \quad 0 \le p \le n-1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_+ &= \sum_i (ie_{2i} + (n-i)e_{2i+1}) \\
H &= \sum_i (ih_{2i} + (n-i)h_{2i+1}).
\end{aligned}
$$

As shown in [9] it is possible to find a Chevalley basis for $osp(2m \pm 1 \mid 2m)$ such that the generators $J_\pm$ and H of the $osp(1|2)$ subalgebra have the same expressions as for $sl(n+1|n)$. For $osp(2m+1|2m)$ it is given by

$$
\begin{aligned}
h_i &= (-1)^i(E_{2m+1-i,2m+1-i} - E_{2m+1+i,2m+1+i} + E_{2m+2-i,2m+2-i} - E_{2m+i,2m+i}) \\
e_i &= (-1)^i(E_{2m+i,2m+1+i} - E_{2m+1-i,2m+2-i}) \\
f_i &= E_{2m+1+i,2m+i} + E_{2m+2-i,2m+1-i}
\end{aligned}
$$

and for $osp(2m-1|2m)$ by

$$
\begin{aligned}
h_i &= (-1)^{i+1} \left(E_{2m-i,2m-i} - E_{2m+i,2m+i} + E_{2m+1-i,2m+1-i} - E_{2m-1+i,2m-1+i} \right) \\
e_i &= (-1)^i \left(E_{2m-i,2m+i} - E_{2m-i,2m+1-i} \right) \\
f_i &= E_{2m+i,2m-1+i} + E_{2m+1-i,2m-i}.
\end{aligned}
$$

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